CHAPTER 7

Analytic Trigonometry

7.3 Double-Angle, Half-Angle, and Product-Sum Formulas

Objectives

- Double-Angle Formulas
- Half-Angle Formulas
- Simplifying Expressions Involving Inverse Trigonometric Functions
- Product-Sum Formulas

Double-Angle, Half-Angle, and Product-Sum Formulas

The identities we consider in this section are consequences of the addition formulas. The **Double-Angle Formulas** allow us to find the values of the trigonometric functions at $2x$ from their values at x.

The **Half-Angle Formulas** relate the values of the trigonometric functions at $\frac{1}{2}x$ to their values at x. The **Product-Sum Formulas** relate products of sines and cosines to sums of sines and cosines.

Double-Angle Formulas

Double-Angle Formulas

The formulas in the following box are immediate consequences of the addition formulas.

DOUBLE-ANGLE FORMULAS

Formula for sine: $\sin 2x = 2 \sin x \cos x$

Formulas for cosine:

$$\begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x \\ &= 1 - 2\sin^2 x \\ &= 2\cos^2 x - 1 \end{aligned}$$

Formula for tangent: $\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$

Example 2 – *A Triple-Angle Formula*

Write $\cos 3x$ in terms of $\cos x$.

Solution:

$\cos 3x = \cos(2x + x)$

$= \cos 2x \cos x - \sin 2x \sin x$ Addition formula

$= (2\cos^2 x - 1)\cos x$
$\quad - (2\sin x \cos x)\sin x$ Double-Angle Formulas

$= 2\cos^3 x - \cos x - 2\sin^2 x \cos x$ Expand

Example 2 – *Solution*

cont'd

$$= 2\cos^3 x - \cos x - 2\cos x\,(1 - \cos^2 x)$$ Pythagorean identity

$$= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x$$ Expand

$$= 4\cos^3 x - 3\cos x$$ Simplify

Double-Angle Formulas

Example 2 shows that $\cos 3x$ can be written as a polynomial of degree 3 in $\cos x$.

The identity $\cos 2x = 2 \cos^2 x - 1$ shows that $\cos 2x$ is a polynomial of degree 2 in $\cos x$.

In fact, for any natural number n, we can write $\cos nx$ as a polynomial in $\cos x$ of degree n.

Half-Angle Formulas

Half-Angle Formulas

The following formulas allow us to write any trigonometric expression involving even powers of sine and cosine in terms of the first power of cosine only.

This technique is important in calculus. The Half-Angle Formulas are immediate consequences of these formulas.

FORMULAS FOR LOWERING POWERS

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x}$$

Example 4 – *Lowering Powers in a Trigonometric Expression*

Express $\sin^2 x \cos^2 x$ in terms of the first power of cosine.

Solution:

We use the formulas for lowering powers repeatedly:

$$\sin^2 x \cos^2 x = \left(\frac{1 - \cos 2x}{2}\right)\left(\frac{1 + \cos 2x}{2}\right)$$

$$= \frac{1 - \cos^2 2x}{4} = \frac{1}{4} - \frac{1}{4}\cos^2 2x$$

$$= \frac{1}{4} - \frac{1}{4}\left(\frac{1 + \cos 4x}{2}\right) = \frac{1}{4} - \frac{1}{8} - \frac{\cos 4x}{8}$$

$$= \frac{1}{8} - \frac{1}{8}\cos 4x = \frac{1}{8}(1 - \cos 4x)$$

Example 4 – *Solution*

cont'd

Another way to obtain this identity is to use the Double-Angle Formula for Sine in the form $\sin x \cos x = \frac{1}{2} \sin 2x$. Thus

$$\sin^2 x \cos^2 x = \frac{1}{4} \sin^2 2x = \frac{1}{4}\left(\frac{1 - \cos 4x}{2}\right)$$

$$= \frac{1}{8}(1 - \cos 4x)$$

Half-Angle Formulas

HALF-ANGLE FORMULAS

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}} \qquad \cos\frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}}$$

$$\tan\frac{u}{2} = \frac{1-\cos u}{\sin u} = \frac{\sin u}{1+\cos u}$$

The choice of the $+$ or $-$ sign depends on the quadrant in which $u/2$ lies.

Example 5 – *Using a Half-Angle Formula*

Find the exact value of sin 22.5°.

Solution:

Since 22.5° is half of 45°, we use the Half-Angle Formula for Sine with *u* = 45°. We choose the + sign because 22.5° is in the first quadrant:

$$\sin\frac{45^\circ}{2} = \sqrt{\frac{1-\cos 45^\circ}{2}} \qquad \text{Half-Angle Formula}$$

$$= \sqrt{\frac{1-\sqrt{2}/2}{2}} \qquad \cos 45^\circ = \sqrt{2}/2$$

Example 5 – *Solution*

cont'd

$$= \sqrt{\frac{2 - \sqrt{2}}{4}}$$ Common denominator

$$= \tfrac{1}{2}\sqrt{2 - \sqrt{2}}$$ Simplify

Evaluating Expressions Involving Inverse Trigonometric Functions

Evaluating Expressions Involving Inverse Trigonometric Functions

Expressions involving trigonometric functions and their inverses arise in calculus. In the next example we illustrate how to evaluate such expressions.

Evaluate $\sin 2\theta$, where $\cos \theta = -\frac{2}{5}$ with θ in Quadrant II.

Solution :

We first sketch the angle θ in standard position with terminal side in Quadrant II as in Figure 2.

Since $\cos \theta = x/r = -\frac{2}{5}$,
we can label a side and the hypotenuse of the triangle in Figure 2.

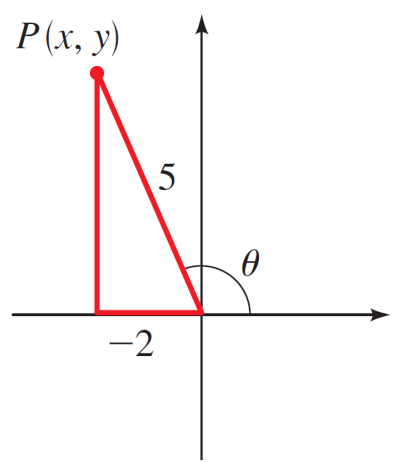

Figure 2

Example 8 – *Solution*

cont'd

$$x^2 + y^2 = r^2$$ Pythagorean Theorem

$$(-2)^2 + y^2 = 5^2$$ $x = -2$, $r = 5$

$$y = \pm\sqrt{21}$$ Solve for y^2

$$y = +\sqrt{21}$$ Because $y > 0$

We can now use the Double-Angle Formula for Sine:

$$\sin 2\theta = 2 \sin \theta \cos \theta$$ Double-Angle Formula

$$= 2\left(\frac{\sqrt{21}}{5}\right)\left(-\frac{2}{5}\right)$$ From the triangle

$$= -\frac{4\sqrt{21}}{25}$$ Simplify

Product-Sum Formulas

Product-Sum Formulas

It is possible to write the product sin u cos v as a sum of trigonometric functions. To see this, consider the addition and subtraction formulas for the sine function:

$$\sin(u + v) = \sin u \cos v + \cos u \sin v$$

$$\sin(u - v) = \sin u \cos v - \cos u \sin v$$

Adding the left- and right-hand sides of these formulas gives

$$\sin(u + v) = \sin(u - v) = 2 \sin u \cos v$$

Product-Sum Formulas

Dividing by 2 gives the formula

$$\sin u \cos v = \tfrac{1}{2}[\sin(u + v) + \sin(u - v)]$$

The other three **Product-to-Sum Formulas** follow from the addition formulas in a similar way.

PRODUCT-TO-SUM FORMULAS

$$\sin u \cos v = \tfrac{1}{2}[\sin(u + v) + \sin(u - v)]$$

$$\cos u \sin v = \tfrac{1}{2}[\sin(u + v) - \sin(u - v)]$$

$$\cos u \cos v = \tfrac{1}{2}[\cos(u + v) + \cos(u - v)]$$

$$\sin u \sin v = \tfrac{1}{2}[\cos(u - v) - \cos(u + v)]$$

Product-Sum Formulas

The Product-to-Sum Formulas can also be used as Sum-to-Product Formulas. This is possible because the right-hand side of each Product-to-Sum Formula is a sum and the left side is a product. For example, if we let

$$u = \frac{x + y}{2} \qquad \text{and} \qquad v = \frac{x - y}{2}$$

in the first Product-to-Sum Formula, we get

$$\sin \frac{x + y}{2} \cos \frac{x - y}{2} = \tfrac{1}{2}(\sin x + \sin y)$$

so

$$\sin x + \sin y = 2 \sin \frac{x + y}{2} \cos \frac{x - y}{2}$$

Product-Sum Formulas

The remaining three of the following **Sum-to-Product Formulas** are obtained in a similar manner.

SUM-TO-PRODUCT FORMULAS

$$\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$$

$$\sin x - \sin y = 2 \cos \frac{x+y}{2} \sin \frac{x-y}{2}$$

$$\cos x + \cos y = 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}$$

$$\cos x - \cos y = -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}$$

Example 11 – *Proving an Identity*

Verify the identity $\dfrac{\sin 3x - \sin x}{\cos 3x + \cos x} = \tan x$.

Solution :

We apply the second Sum-to-Product Formula to the numerator and the third formula to the denominator:

$$\text{LHS} = \frac{\sin 3x - \sin x}{\cos 3x + \cos x} = \frac{2 \cos \dfrac{3x + x}{2} \sin \dfrac{3x - x}{2}}{2 \cos \dfrac{3x + x}{2} \cos \dfrac{3x - x}{2}}$$

Sum-to-Product Formulas

Example 11 – *Solution*

cont'd

$$= \frac{2\cos 2x \sin x}{2\cos 2x \cos x}$$ Simplify

$$= \frac{\sin x}{\cos x}$$ Cancel

$$= \tan x$$

$$= \text{RHS}$$